Hanging Pictures Activity – Question Sheet

You have an alphabet of letters $\vec{A}, \overleftarrow{A}, \vec{B}, \overleftarrow{B}, \vec{C}, \overleftarrow{C}$ which satisfy a special rule.

Whenever a letter like $\vec{A}$ is next to $\overleftarrow{A}$, these letters annihilate each other!

A word is *self-destructive* if it completely annihilates itself, like $\vec{C}\vec{B}\overleftarrow{B}\overleftarrow{C}$

Try to answer the following questions:

Q1: Find a word with 6 letters which is self-destructive.

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Q2: Is it possible to find a word with 7 letters which is self-destructive?

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Q3: Find a stable word which only becomes self-destructive when every copy of the letter B is removed.

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Q4: Find a word using the letters $\vec{A}, \overleftarrow{A}, \vec{B}, \overleftarrow{B}$ satisfying the following three conditions:
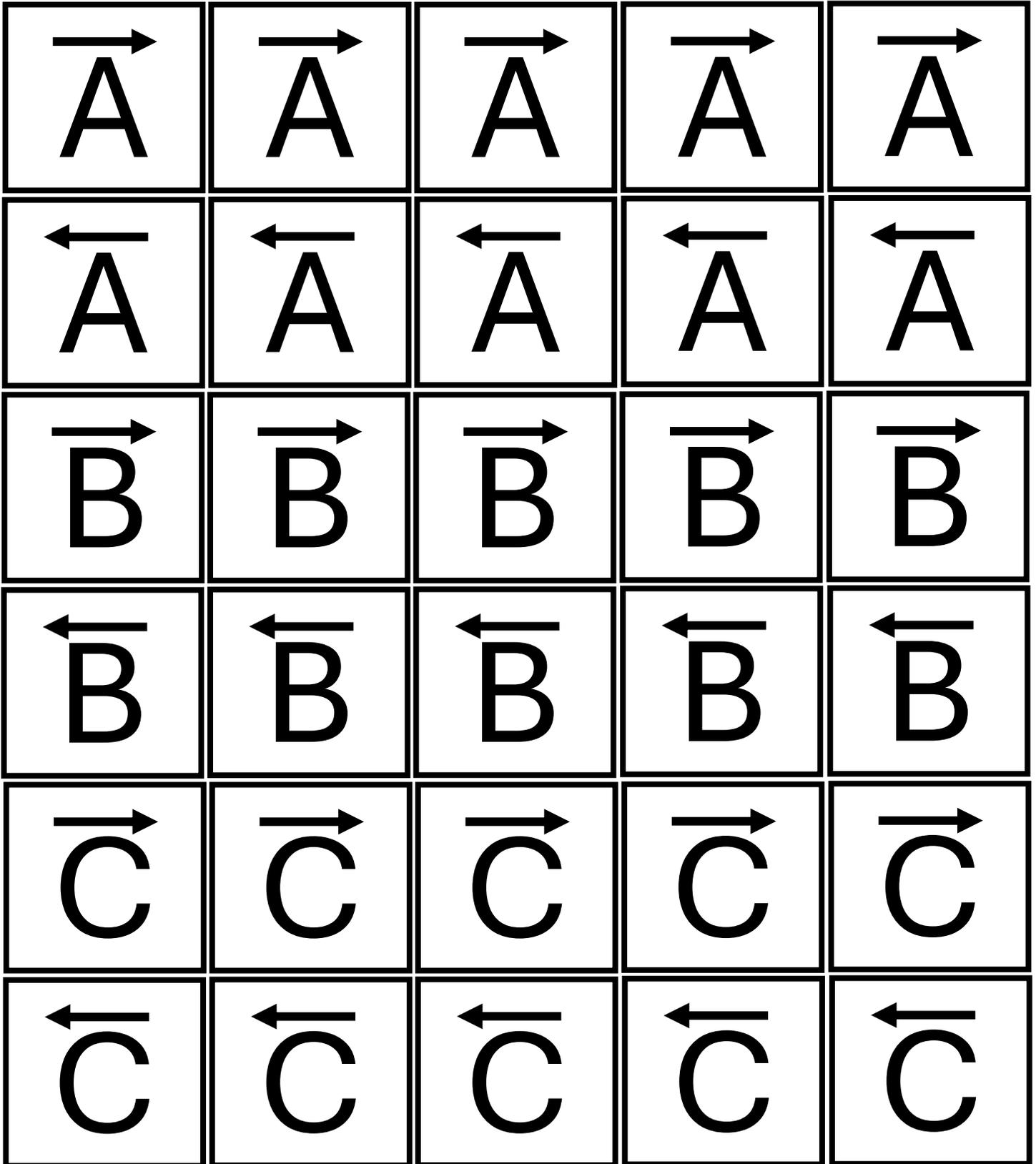
    a) It is stable (so no annihilating pairs)
    b) It becomes self-destructive when removing every A.
    c) It becomes self-destructive when removing every B.

Bonus: Find the shortest word that satisfies these properties.

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Q5: Repeat **Q4** but now with all of the letters That is, find a stable word which becomes self-destructive upon removing the As, the Bs or the Cs

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Hanging Pictures Activity – Letter Cut-Out Sheet

This is a printable sheet that can be used for cutting out the letters to help learners solve the questions. You should print out one copy of this sheet per group.

$\vec{A}$	$\vec{A}$	$\vec{A}$	$\vec{A}$	$\vec{A}$
$\overleftarrow{A}$	$\overleftarrow{A}$	$\overleftarrow{A}$	$\overleftarrow{A}$	$\overleftarrow{A}$
$\vec{B}$	$\vec{B}$	$\vec{B}$	$\vec{B}$	$\vec{B}$
$\overleftarrow{B}$	$\overleftarrow{B}$	$\overleftarrow{B}$	$\overleftarrow{B}$	$\overleftarrow{B}$
$\vec{C}$	$\vec{C}$	$\vec{C}$	$\vec{C}$	$\vec{C}$
$\overleftarrow{C}$	$\overleftarrow{C}$	$\overleftarrow{C}$	$\overleftarrow{C}$	$\overleftarrow{C}$

Hanging Pictures Outreach Activity – Instructor Activity Sheet
Fern Gossow

<u>Audience</u>: Groups of size 1-12 (large groups should be divided up into 3-4) of ages 10+.

<u>Time</u>: 15-30 minutes. Can be adjusted depending on how much time is spent on each of the problems.

<u>Materials</u>: String/rope, weight to attach to string, board with removable tacks, nails or hooks, letter cards sorted and stacked, instructor activity sheet, learner activity sheets with puzzles.

<u>Activity Overview</u>:

Stage 1: Introducing the Problem (5 min)

Hanging a painting by $\vec{A}\vec{B}\overleftarrow{A}\overleftarrow{B}$

- Hang the weight on two nails in front of the audience as shown.
- Ask everybody to make a prediction about what will happen to the painting if either nail is removed.
- Remove a nail. The weight will fall! Hang it up the same way and remove the other nail to show the same behavior.
- Ask the audience for an explanation of this phenomenon (but don't reveal any answers yet).
- Introduce the three-nail problem: can we hang a painting on three nails so that removing any one of them causes the paining to fall? To find a solution, we want to come up with a new way of looking at this system.
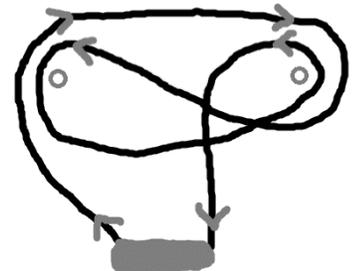
Stage 2: Abstracting the system (5-15 min)

- Introduce the alphabet of letters $\vec{A}, \overleftarrow{A}, \vec{B}, \overleftarrow{B}, \vec{C}, \overleftarrow{C}$. Each letter and its pair are like matter and anti-matter, if they are sitting next to each other in a word then they annihilate each other.
- Explain what happens to the word $\vec{A}\vec{B}\overleftarrow{B}\overleftarrow{A}\vec{C}\overleftarrow{C}$. We call a word *unstable* if it has an annihilating pair, and *self-destructive* if it completely annihilates itself.
- Give out the list of questions to each group. Have them spend time trying to come up with words that satisfy certain properties. Walk around to answer questions, get groups to demonstrate their solutions, and ensure people within each group are all getting a chance to contribute.
- After some time, ask each group to demonstrate the solution to each problem.

Stage 3: Solving the problem (5 min)

- Ask the learners if they can imagine a connection between these letters and the hanging-problem.
- After some time, reveal the answer:
  - Label the nails by the letters. Hang the weight by reading the word, moving over the top of each nail in the direction given by the word.
  - Any annihilating pair can be slipped off (consider hanging the weight using $\vec{A}\overleftarrow{A}$. So hanging a painting by a self-destructive word causes the painting to fall off!
  - Removing all instances of A in a word corresponds to removing the nail A, since we simply ignore how we hung the weight on this nail.
  - In particular, the solution for the last problem on the student activity sheet gives a solution for hanging the painting in the three-nail case.
- Have one of the learners hang up the painting according to their solution. Watch carefully to ensure that they do it correctly. Have a different learner choose one of the nails to remove, and hopefully the weight will fall! Repeat this experiment for each nail.

Remember to be excited and prompt the learners to make lots of predictions and use their imagination!

Hanging Pictures Activity – Answer Sheet

This sheet provides answers to the questions sheet, as well as some additional information for explaining the mathematics behind this problem to learners. The source of explanation and answers for the questions should generally occur in the order:

> (1) From other learners who were also thinking about the problem for the first time.
> (2) From the interpreter, relating the answer to their specific questions and ideas.
> (3) Directly from this sheet if necessary.

Q1: There are many answers here, such as $\vec{A}\overleftarrow{A}\vec{B}\overleftarrow{B}\vec{C}\overleftarrow{C}$, $\vec{A}\overleftarrow{A}\vec{A}\overleftarrow{A}\vec{A}\overleftarrow{A}$ or $\overleftarrow{A}\vec{B}\vec{C}\overleftarrow{C}\overleftarrow{B}\vec{A}$

Q2: No. Since annihilations happens in pairs, every self-destructive word must have even length.

Q3: There are many solutions, such as $\vec{A}\vec{B}\overleftarrow{A}$, $\vec{B}\vec{C}\vec{B}\overleftarrow{C}\overleftarrow{B}\vec{B}$ or $\vec{B}\vec{A}\overleftarrow{A}\overleftarrow{B}$. Mathematicians would argue that the words $\vec{A}\overleftarrow{A}$, $\vec{B}$ and even the empty word " " are solutions, which are sometimes called "trivial".

Q4: The shortest possible solutions are four letters long: $\vec{A}\vec{B}\overleftarrow{A}\overleftarrow{B}$, $\vec{B}\vec{A}\overleftarrow{B}\overleftarrow{A}$, $\overleftarrow{A}\vec{B}\vec{A}\overleftarrow{B}$, $\vec{B}\overleftarrow{A}\overleftarrow{B}\vec{A}$ and so on.

Q5: One way to do this is to start with your solution to Q4, add on $\vec{C}$, add the word which annihilates your solution, and then add $\overleftarrow{C}$.

For example: $\vec{A}\vec{B}\overleftarrow{A}\overleftarrow{B}\vec{C}\vec{B}\vec{A}\overleftarrow{B}\overleftarrow{A}\overleftarrow{C}$.

This gives the shortest possible solution, which has 10 letters. The picture shows this hanging.
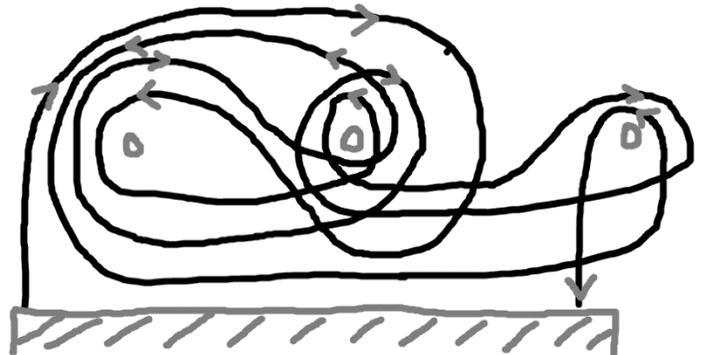
Mathematical explanation of the activity:

The string serves as a model for a *loop* in a *topological space*. In this case, the space we are working with is the 2-dimensional plane $\mathbb{R}^2$ with some points removed. (We think of nails as removing points because our string cannot pass through these points.)

Every space has a *fundamental group*, which is a description of the loops that are possible to create in that space. Two loops are called *homotopic* if one can be deformed into the other within the space. For example, in the 2-dimensional plane $\mathbb{R}^2$, all loops are homotopic.

The fundamental group of our space $\mathbb{R}^2 \setminus \{x_1, x_2\}$ (the plane with two points removed, in the case of two nails) is the *free group on two generators*, denoted as F_2 or $\mathbb{Z} * \mathbb{Z}$. What this means is that every loop can be described as a word in the letters $\vec{A}$, $\overleftarrow{A}$, $\vec{B}$, and $\overleftarrow{B}$, where $\vec{A}$ cancels with $\overleftarrow{A}$ and $\vec{B}$ with $\overleftarrow{B}$.

The translation between the words and the strings is as follows: each time we encounter the letter $\vec{A}$, hook the string around the first nail so that it goes rightward over the top of the nail. We do the same for $\overleftarrow{A}$, but going leftwards. The letter B corresponds to the second nail.

Note that hanging the string according to $\vec{A}\overleftarrow{A}$ or $\overleftarrow{A}\vec{A}$ would mean the string can freely fall, and so these letter pairs must cancel each other out. In particular, a word is *self-destructive* precisely when hanging the painting by this word would cause the string to fall. Removing all copies of a particular letter corresponds to removing the associated nail. Hence, finding a hanging that falls under removing either nail is the same as finding a word which becomes self-destructive after removing either letter!